Not Quite Yet a Hazy Limbo of Mystery: Intuition in Russell's *An Essay on the Foundations of Geometry*

Tyke Nunez

University of South Carolina,

United States of America

AN16@mailbox.sc.edu

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Abstract

I argue that in Bertrand Russell's *An Essay on the Foundations of Geometry* (1897), his forms of externality serve the same fundamental role in grounding the possibility of geometry that Immanuel Kant's forms of intuition serve in grounding geometry in his critical philosophy. Specifically, both provide knowledge of bare numerical difference, where we have no concept of this difference. Because geometry deals with such conceptually homogeneous magnitudes and their composition on both accounts, on both forms of intuition or externality (respectively) are at its foundation.

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§1 – Introduction

According to a venerable tradition in mathematics that traces its roots back through Kant to Euclid, Eudoxus, and Aristotle, geometry was the preeminent mathematical science, fundamentally grounded in space. Bertrand Russell's 1903 *The Principles of Mathematics* is often taken to be the death knell for this tradition, most notably by Russell himself. Five years earlier, however, in his 1897 *Essay on the Foundations of Geometry* (henceforth: *EFG*/*Foundations*), I shall argue that Russell defended a neo-Kantian view that falls squarely within this tradition.

Recently, scholars of Kant like Michael Friedman (1992) or Daniel Sutherland (2021), and philosophers of mathematics like Kenneth Menders (2008) and Charles Parsons (2008), have presented a reassessment of Kant's philosophy of mathematics. Against the various attacks on the Kantian use of the intuition of space as a foundation for mathematics, which had become commonplace by the end of the 19th century, these recent philosophers have made the case that intuition has, on views like Kant's, a quite exact role to play in grounding mathematics.

We should applaud these efforts. Philosophically, although I will not be arguing for this here, a great deal was lost with the advent of modern logic. It builds in that logic has a numerable collection of individuals available to it and turns logic into the study of number and numerical relations.[1] It thereby collapses Kant's distinction between logic, as the most abstract branch of philosophy, and mathematics—the queen and king of the sciences—and thereby collapses Kant's system of sciences into a flat linguistic structure. It also shunts, by fiat, the logical version of the problem of the one and the many, which asks how thought can be both one and yet treat diversity, a problem which had been at the core of the philosophical agenda since the pre-Socratics. In contrast, this problem remains at the core of Kant's philosophy, because he separates sensibility and intuition from the intellect and concepts, as the sources in our knowledge of multiplicity and unity, respectively. While modern mathematical logic has been tremendously fruitful and a wholesale return to syllogistic logic would be inappropriate

[1] As Tarski (1966) elegantly observes.

and impossible, these serious philosophical losses have not been widely appreciated. My hope is that these can be mitigated through investigating conceptions of the mathematical sciences that acknowledge their grounding in a non-intellectual source of knowledge of multiplicity.

For those interested in the tradition that takes space to lie at the foundation of mathematics and its prospects, Russell's very early view in *Foundations* is worth attention. This is because he is, as I will argue here, staying true to the insights in the philosophy of mathematics at the core of Kant's account. Furthermore, in this very early view of Russell's, I think we find the *only* serious attempt to hold onto a recognizably *exact* successor to Kantian spatial intuition, while accommodating 19th century projective developments in geometry.[2] In this sense, despite the mathematical and philosophical problems with the *Foundations* view, examining it should be fruitful for reflecting on how Kant might also have dealt with many of the 19th century developments in geometry.

In this essay I present why exactly Russell's *Foundations* view of geometry and space is a successor to Kant's. This argument will hinge on why Russell's 'forms of externality' are properly understood to be a successor to Kantian intuition. We will see that like Kant's forms of intuition, Russell's forms of externality are a non-intellectual source of knowledge, and this knowledge lies at the foundation of Geometry. More specifically, we will see that what is essential to both Kant's form of spatial intuition and Russell's forms of externality is that they provide knowledge of bare numerical difference, where we have no concept of this difference. While the intellect on both views affords knowledge of qualities and qualitative differences, both Kant and Russell think we can imagine two qualitatively identical things that we can nonetheless distinguish, like, say two qualitatively identical raindrops, or two one-inch cubes. Both Russell and Kant take geometry to study such qualitatively identical, yet distinct spatial extents, along with their composition. As a result, they both take geometry to have at its foundation such non-intellectual knowledge of bare numerical difference, where we have no concept of this difference.

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[2] On this point compare Griffin (1991, p. 104). Still, in some respects Herman Helmholtz and Joseph Delboeuf count as mid-century antecedents, while two of Russell's contemporaries who influenced him are Arthur Hannequin (1895) and Louis Couturat (1896). Russell reads their books, respectively, in March and August of 1896. Hannequin stresses the incommensurability of the continuous quantities of intuition with the discrete quantities of the intellect and argues that the concept of the atom is the proper tool for science to approximate the reduction of the former to the latter (e.g., p. 10-11, 17, 65, 69, 273). Couturat stresses the importance of the homogeneity of space and of numerically distinct but qualitatively (conceptually) identical geometric quantities drawn from intuition, but he takes continuity to be an idea constructed by reason, not a feature of the form of outer intuition (see, esp. Bk. IV, ch. 2-3).

Although forms of externality are recognizably the successor to Kantian intuition because they provide knowledge of bare numerical difference, the view of *Foundations* also has many Leibnizian or British Idealist innovations that a stricter Kantian should reject, even after considering the advent of projective geometry. In *Foundations* Russell rejects Kant's distinction between analytic and synthetic judgments, he tries to strip out what he takes to be psychological or metaphysical rather than epistemic features of Kant's view (e.g., *EFG*, §5, p. 3-4; §193, p. 187), and he modifies Kant's division between intuitions and concepts.[3] Although important, these non-Kantian elements of Russell's view will not be my focus. I will, however, touch on them in the final section.

I will begin by presenting how a proper understanding of forms of externality as a non-intellectual source of knowledge should reorient the discussion of Russell's early philosophy of mathematics (§2). I will then turn to Kant, the importance of the homogeneity of space on his account, why this homogeneity is linked to the way in which a form of intuition like space can grant knowledge of bare numerical difference, and how he sees this as foundational to mathematics (§3). Next, I will return to Russell and how on his view forms of externality provide knowledge of bare numerical difference (§4), the importance of homogeneity (§5), and how this is essential to the subject matter of both projective and metric geometry (§6). Finally, I will examine the logicist sounding rhetoric of *Foundations* and why we should not take this to be an endorsement of logicism (§7).

§2 – A non-intellectual source of knowledge?

Since Russell was once an idealist, it is unsurprising that he once took something like Kantian intuition to lie at the basis of geometry.[4] Among scholars of *Foundations* it is also well known that he conceived of the book's main argument as a 'transcendental proof' or 'deduction' that he modeled on Kant's own transcendental deductions, whereby he attempts to show that the axioms of projective geometry and the axioms common to all metric

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[3] Russell treats the exact status of 'forms of externality' and how they are not quite 'sensational', 'conceptual', or 'intuitional', in some detail at the end of *Foundations*. It is primarily in reflecting on this discussion that I think one can see how exactly Russell is modifying Kant's account of the division between concept and intuition. As a result, it is important that I am only claiming that forms of externality are a non-intellectual source of knowledge, not that they are an intuitional source of knowledge. While there is not space here to delve into these complex considerations, I hope to treat the topic in future work.

[4] Indeed, Coffa (1981), Hylton (1990, ch. 4), Griffin (1991, 2012, 2022), Shieh (2019, ch. 6), and even to a degree Heis (2017) all accept this.

geometries are necessary conditions on the possibility of experience.[5] If these continuities between the views of Kant and Russell in *Foundations* are well known, then what does the present essay contribute to the discussion?

Although prior interpreters take Russell's forms of externality to be something like Kantian intuition, I do not think that they have understood either *why* Russell takes his forms of externality to provide a non-intellectual source of knowledge or *why* he takes such a non-intellectual source of knowledge to be essential to geometry. As a result, by my lights, they have missed the core of the *Foundations* view and they have missed how the fundamental issue driving Russell's thinking in the early period is his shifting position on whether forms of externality can be an adequate foundation for geometrical knowledge (and even all of mathematics). This has led to two further connected problems. The first is that even the best interpreters of this period largely treat Russell's views prior to January of 1899 as though they were homogenous. The second is that interpreters have sometimes thought that Russell's pre-1899 views are much closer to his later logicist view of *Principles* than they in fact are.

Let me say a little bit to substantiate these two claims. In telling the story of how Russell arrives at his logicism of *Principles*, a lot of attention has rightly been paid to the August 1900 encounter with Peano at the Paris International Congress of Philosophy and the revolution in Russell's thinking at the end of 1898 evident in 'The Classification of Relations' (1899). Before this the distinctions are thought to be rather minor. Once we see that the key issue prior to 1899, however, is whether forms of externality can be a genuine non-intellectual source of knowledge we can see that Russell holds at least three radically distinct and mutually conflicting positions in the period up to 1899.[6] The first is the view up to and including *Foundations.* On it forms of externality are taken to be a source of knowledge, and by positing intrinsically identical atoms Russell seems to think he can escape the antinomies that afflict geometry through the move to kinematics. The second position—represented by his *Mind* paper, 'On the relations of number and quantity' (1897b)—is held from sometime in 1897, up through the beginning of 1898. On it forms of externality and intuition are rejected as a source of knowledge, because he holds that the continuum is afflicted with antinomies, but he can find no suitable replacement. The third position is the one found after 'On Quantity and Allied Conceptions' (1898a), through 'Analysis of Mathematical Reasoning' (1898c), and up to the

[5] Hylton (1990, ch. 4, esp. 81-83), Griffin (1991, §4.3-4.6), Shieh (2019, ch. §6.2), and Heis (2017, §2).
[6] See my (forthcoming) for more discussion of these periods.

revolution in his thinking at the end of 1898. In this period Russell's view becomes more Leibnizian because he no longer takes points to be intrinsically identical, but treats them as more like monads, where their position is a unique intrinsic quality that differentiates them from all other points (1898b, p. 311, 320).

We can get a sense of the important differences between these three periods by noting the shifting place of what Russell calls in 1898 'the contradiction of relativity' and how this contrasts with the place that Nicolas Griffin (1991, 2012, 2022) attributes to it. Griffin, along with Peter Hylton (1990, ch. 4), has long emphasized the importance of Russell's rejection of the doctrine of internal relations and his adoption of Platonic Atomism at the end of 1898 for the development of his logicism. For this Griffin has rightly stressed the importance of the 'contradiction of relativity', which arises wherever we have 'a conception of difference with no difference of conception' (1897b, p. 81). Griffin, however, takes this contradiction to be equally pressing in early 1897 in *Foundations*.[7] I hold, on the contrary, that it is not. Griffin and others fail to see this because they fail to see the fundamental role played in *Foundations* by forms of externality. Although shortly after the appearance of *Foundations* Russell comes to take knowledge of bare numerical difference, where we have no concept of this difference, to be problematic and to generate a 'contradiction' because there can be no intellectual or conceptual source of it, in *Foundations* he still sees no problem with a second, non-intellectual source of this knowledge—namely his forms of externality. As a result, it is not, as Griffin maintains, that Russell just hadn't explicitly formulated the contradiction of relativity yet in *Foundations* (2022, p. 11; 2012, p. 1; compare 1991, p. 187, 189, 182-183), rather it is that the kind of knowledge that came to seem problematic on the 1898 Leibnizian view, was not problematic on the 1897 Kantian one.[8]

[7] For example, Griffin claims that the contradiction of relativity 'had first appeared in *An Essay on the Foundations of Geometry* (1897) as the 'antinomy of the point:' namely that, while each point is distinct from every other, all are exactly alike' (2022, p. 11; 2012, p. 1). Both Heis (2017, p. 327) and Hylton (1991, p. 83) also make remarks that suggest that they endorse this position.

[8] This is related to Griffin's not seeing that, as I argue in (forthcoming), Russell rejects the doctrine of internal relations in *Foundations* within geometry, although he comes to endorse it there in his more Leibnizian period. According to this doctrine, all relations are grounded in and even determined by intrinsic properties of the terms related. This doctrine is related to the contradiction of relativity because a difference of conception is a difference in the intrinsic properties of the related terms, and so if the doctrine held without leading to contradiction, then for every conception of difference, there would be a difference in conception.

In my (forthcoming) I focus on the third of Russell's antinomies about spatial figures and relations. But one bit of evidence from *Foundations* that might seem to speak in favor of an endorsement of the doctrine of internal relations in geometry is from Russell's introduction of his first antinomy, the antinomy of the point:

The second problem is evident in a recent provocative and stimulating essay by Jeremey Heis (2017). According to Heis, Russell's position in 1898 counts as logicist because it takes mathematics to be grounded entirely in logic and conceptual knowledge.[9] Heis has argued that intuition only ever had a small role to play for Russell, and that even in *Foundations* pure mathematics could be deduced from the intellect or the laws of thought alone (2017, p. 309). According to Heis, Russell thinks that from the laws of thought we can deduce that there must be some form of externality, which allows us to be conscious in perception of numerically diverse things (2017, p. 313-314). As a result, he holds that intuition or sensation exhibits the reality of a particular metrical geometry, but that mathematics is otherwise independent of this exhibition. For this reason, according to Heis, Russell's forms of externality are grounded ultimately in the intellect, and do not constitute a second essentially separate source for knowledge. I will argue, contrary to this interpretation, that Russell's forms of externality serve as the bedrock of geometrical knowledge and are such an essentially separate, non-intellectual source of knowledge. In the final section (§7), I will return to evaluate Heis's claim that although Russell's position in *Foundations* is not strictly logicist, it still counts as a 'nonstandard sort of 'logicism', which employs a conception of logic as transcendental' (2017, p. 304).

§3 – Homogeneity in Kant and the role of intuition in mathematics

To see why forms of externality have an essential role to play in Russell's *Foundations* account, and how this role is similar to the role of intuition on Kant's account, it will help to first see why Kant maintains that intuition forms the foundation of mathematical and

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> Though the parts of space are intuitively distinguished, no conception is adequate to differentiate them. Hence arises a vain search for elements, by which the differentiation could be accomplished, and for a whole, of which the parts of space are to be components. Thus, we get the point, or zero extension, as the spatial element, and an infinite regress or a vicious circle in the search for a whole. (*EFG*, §195, p. 188)

Here it can sound like Russell endorses the doctrine of internal relations and thus the contradiction of relativity. This is because it is our inability to find a concept that is adequate for distinguishing the parts of space that leads us to points, and points are contradictory because they are both supposed to be the basic element of space out of which it is composed and something extensionless, the composition of which could never yield a spatial extent.

Notice, however, that Russell is here claiming that the parts of space are 'intuitively distinguished', and so is allowing that we have a non-intellectual source of knowledge of their distinctness. The antinomy stems not from the qualitative or conceptual identity of points—which is a feature they share with one-inch cubes—but from the fact that they are extensionless. Thus, while we might seek a concept for a non-conceptual distinction, at this stage Russell doesn't take such a distinction itself to be problematic.

[9] Sanford Shieh (2019, ch. 6) has also argued for a similar conclusion from a different direction.

geometrical knowledge. Here I will be indebted to Daniel Sutherland, especially chapter 7 of his new book *Kant's Mathematical World* (2021).

Following Sutherland, I'd like to begin with Kant's conception of the homogeneity of intuition. Euclid relies on a conception of the homogeneity of spatial figures and magnitudes, but neither Euclid nor his commentators give an analysis of homogeneity (Sutherland, 2021, p. 198). Kant attempts to fill this lacuna by explaining a magnitude 'as a homogeneous manifold in intuition in general' (B203), where this notion of homogeneity adverts to the etymology of the term: sameness of genus.[10] As Sutherland puts it, 'two concepts or things are homogeneous with respect to a concept if they both fall under that concept' (2021, p. 199). For example, the concept of a dog and the concept of a horse are homogeneous with respect to the concept mammal. This logical notion of homogeneity comes in degrees. Concepts, for Kant, are ordered into genus-species hierarchies. And concepts or objects can be more or less logically homogeneous depending on where they sit in relation to one another in such a hierarchy. For example, <dog> and <horse> are more homogeneous with each other than with <alligator> because in the hierarchy they are closer to their lowest common concept, say <mammal>, than they are to their lowest common concept with, say, <alligator>.

For our purposes, what will be important is not this relative notion of homogeneity, but a narrower, stricter notion. To exhibit this kind of strict homogeneity, which is required for any *quantum* or magnitude*,* like a spatial figure, 'things' must be 'from one and the same genus' (*Met-V,* 1794-5, 29:991). That is, they must be the same in kind, and fall under all of the same concepts. In describing this feature of quantities or *magnitudes*, Kant relies on a contrast with quality:

> Quality differs from quantity [*Quantität*] in that, and to the extent that, the [*former*] indicates something in the same object which is inhomogeneous [*ungleichartiges*] with regard to other determinations found in it. Hence, quality is that determination of a thing according to which whatever is specifically different finds itself under the same genus, and can be distinguished from it. This is heterogeneous [*heterogen*] in opposition to that which is not specifically different, or to the homogeneous [*homogen*]. (*Met-V,* 1794-5, 29:992)

On Kant's view, then, qualities are ways in which two things are specifically different. They are features that indicate a respect in which two things are heterogenous. Quantities of the same kind, however, do not allow for specific differences. In this sense they are qualitatively identical, or qualitatively homogeneous. For this reason, quantities are grounded in a kind of

[10] Kant's Critique of Pure Reason is cited using the standard A and B edition numbering. Kant's other works are cited by volume and page number of the Akademie edition, together with the standard *Kantian Review* abbreviations for the work.

maximum or complete logical homogeneity, or what I will call, following Sutherland, 'strict homogeneity' (2021, §7.3, p. 201).

Kant explains this notion of strict homogeneity in terms of specific identity along with numerical difference. For example, he claims that 'two drops of water on two needle points are numerically different and specifically identical' (*Met-V*, 28:422; A263-264/B319-320). Or 'Homogeneity is specific identity with numerical diversity [*numerischen Diversitaet*], and a *quantum* consists of homogeneous parts [*partibus homogeneis*]' (*Met-Schön*, 28:504, late 1780s). Bare numerical difference, then, cannot be represented through concepts alone. This is because concepts represent specific differences. That is, they represent a division into species that differentiates two classes of members of a genus. Because in this case there are no specific differences with which to distinguish what is represented, concepts on Kant's account cannot represent this kind of bare numerical difference.

In contrast, intuition, and the *a priori* forms of space and time, contain a multiplicity or manifold of specifically identical, yet numerically distinct parts. That is, if we take two sixty second stretches, or two one-inch cubes, and consider not what fills them—which might be qualitatively distinct—but only their *a priori* form, or that which is in each case filled, then we will have examples of the sort of qualitative identity with numerical diversity that Kant has in mind. A basic task of intuition is to provide this kind of knowledge of bare numerical difference, where there is no concept of the difference.

This is connected to Kant's critique of Leibniz in the Amphiboly. Leibniz thinks that all knowledge is intellectual. He thus thinks that wherever there is a distinction between two individuals, there is a specific difference between them, and this difference can be represented conceptually. Kant, however, thinks that because we can imagine qualitatively and quantitatively identical regions in space and time, we are presented with numerical difference where there is no specific difference. Kant agrees with Leibniz that the identity of indiscernibles holds for intelligible objects, where the only distinctions between them are specific differences that the intellect can form a concept of, without the aid of sensible intuition. He can also agree with Leibniz that the identity of indiscernibles would hold for objects of the sensible world, if the only way they could be known was through the intellect and its concepts. But because sensible objects can be known through intuition, we can know that two specifically identical individuals are distinct, and so for them the identity of indiscernables fails (A264/B320). The root of Leibniz's problem, then, according to Kant, lies in the fact that he does not recognize the intellect and sensation as sources of distinct kinds of knowledge, but mistakes intuitions for confused concepts.

Here, of course, is not the place to adjudicate the fairness of Kant's critique, but for the purposes of understanding Kant's account we will need a sense of the qualitative differences that still matter for geometry. Within Geometry, of course, the usual kinds of qualitative features like color or warmth are not relevant and are abstracted away from. Nonetheless, like Euclid, Kant thinks of magnitudes or figures as differentiated in kind by their number of dimensions. In this way, for both Kant and Euclid, lines are only homogeneous with lines, angles with angles, areas with areas, and volumes with volumes. A line counts as a magnitude for Kant, then, because the manifold parts of the line are strictly homogeneous with one another, or an area contains smaller areas that are homogeneous with one another, etc. (Sutherland 2021, p. 208). Moreover, two distinct lines are homogeneous because the parts contained in one are homogeneous with the parts contained in the other, at least as far as they are considered by geometry. The same holds of areas and volumes. Nonetheless, a line is one dimensional, and this is something it shares with time, but according to Kant, times are not homogeneous with lines, because there is a qualitative difference between times and spaces that distinguishes them. Setting these kinds of cases aside, however, within geometry a difference in dimensionality is a difference in quality, and this is the kind of difference in quality that counts when it comes to the composition of homogeneous manifolds.[11]

Why then does Kant take the presentation of bare numerical difference in intuition to be foundational to mathematics? One part of this account, for Kant, is that intuitions are singular representations, while concepts are general representations, although they can be used singularly (A320/B376-377). Connected to this, as we add more features to a concept and specify it further, it will be true of fewer and fewer objects, as it becomes ever more determinate. Conversely, conceptually, if we are to represent more objects our concepts must become more abstract. This is not true for intuition. As Kant puts it, 'for a part of space, even

[11] I have been arguing that two things are qualitatively identical for Kant when there is no specific difference that distinguishes them. Although this will continue to be our focus, there is a second notion of quality in Kant that is also relevant in geometrical contexts. Qualities in this second sense are features that can be cognized distinctly without comparison to something else, like that a line is straight or curved, or even that something is a magnitude (*quantum*) (*Met-V*, 29:992-993). These qualities contrast with quantity (*quantitas*) as answering the question 'how much?' The quantity of something in this sense is a measure of it, and determinate cognition of such quantity always requires comparison with a unit, thus with something outside the thing measured.

though it might be completely similar and equal to another, is nevertheless outside of it, and is on that account a different part from that which is added to it in order to constitute a larger space' (A264/B320). Thus, we can add more intuition to our intuition, thereby representing more, without making the representations in question more abstract. This kind of combination of strictly homogeneous parts into ever larger wholes, however, cannot be represented by concepts no matter how they are combined, and it is this kind of combination—what Kant calls 'composition'—that he takes to be characteristic of 'everything that can be considered mathematically' (B201n). Kant makes this explicit when he claims that 'according to mere concepts of the understanding, it is a contradiction to think of two things outside of each other that are nevertheless fully identical in respect of all their inner determinations (of quantity and quality)'. This is because 'it is always one and the same thing thought twice (numerically one)' (*Progress*, 20:280; compare A263/B319, A282/B338). Kant's thought is that if we attempted to use only, say, the concept of a one-inch cube to represent the composition of homogeneous parts of space into a larger space, we would represent the cubes of space as instances of the same concept. Then, however, so long as we excluded the use of intuition, the instances would be indistinguishable and hence identical. Thus, we would only succeed in picking out the same thing twice, not in composing two spaces into a new space. As a result, because the task of mathematics is fundamentally to compose and decompose diverse but strictly homogeneous elements, and concepts alone cannot represent such bare numerical diversity, let alone compose such homogeneous parts into a new whole, concepts alone are inadequate for grounding mathematics.

§4 – Forms of externality and multiplicity

To see why forms of externality and the intellect are ultimately two independent sources of knowledge on the *Foundations* account, we need to understand how Kant's forms of intuition and Russell's forms of externality serve the same fundamental role for both. Over the next three sections we will elaborate three dimensions of Russell's forms of externality that are shared with Kant's forms of intuition. (i) Forms of externality ground a diverse manifold of qualitatively or intrinsically identical positions. (ii) These intrinsically identical positions are not merely a heap but stand in ordered relations to one another. Thus, (iii) because forms of externality allow for the distinction of qualitatively identical but numerically distinct spatial figures, they are an essential foundation for geometry on the *Foundations* account.

(i) Today we tend to take for granted that there are a plurality of things and that thought has available to it a diversity or multiplicity of discrete entities to think about. Perhaps surprisingly, both Kant and early Russell did not.[12] Rather, in line with what we just saw, Kant took cognition of diversity or multiplicity to be grounded in the forms of intuition—space and time—and Russell's closely related notion of a form of externality serves this same role. Indeed, it is because something serves this role that it counts as a form of externality, since according to Russell, such a form is whatever allows us to immediately differentiate a diverse manifold of positions.

That Russell appreciates this convergence is evident in Russell's discussion of Kant's first argument about space from §2 of the Transcendental Aesthetic. According to Kant:

> Space is not an empirical concept that has been drawn from outer experiences. For in order for certain sensations to be related to something outside me (i.e., to something in another place in space from that in which I find myself), thus in order for me to represent them as outside and next to one another, thus not merely as different but as in different places, the representation of space must already be their ground. Thus the representation of space cannot be obtained from the relations of outer appearance through experience, but this outer experience is itself first possible only through this representation. (B38)

The representation of space, according to Kant, grounds the representation of different places in space. The representation of space is not something that we discover empirically, by first encountering things in space and representing them. Rather, representing things as outside and next to one another, and thereby representing their spatial features at all, depends on a *prior* representation of space.

Commenting on Kant's argument Russell claims that 'the proper function of space is to distinguish between different presented things', and argues that Kant's first argument about space comes down to

> the following: consciousness of a world of mutually external things demands, in presentations, a cognitive but non-inferential element leading to the discrimination of the objects presented. This element must be non-inferential, for from whatever number or combination of presentations, which did not of themselves demand diversity in their objects, I could never be led to infer the mutual externality of their objects. (*EFG*, §58, p. 61)[13]

Russell goes on to claim that 'the *logical* scope' of this argument 'extends, not to Euclidian space, but only to any form of externality which could exist intuitively, and permit knowledge,

[12] For example, at the end of the 1897 manuscript 'Why do we Regard Time, but not Space, as Necessarily a Plenum?' Russell identifies the issue of monism vs pluralism (or monadism) as 'the most fundamental question of metaphysics' (1897c, p. 97).

[13] 'Presentation' is Russell's translation of '*Vorstellung*', which is a standard piece of Kant jargon. These days it is usually translated as 'representation'. Russell is using it to stay neutral between sensation (*Empfindung*) and intuition (*Anschauung*). (Thank you to a referee for asking me to comment on Russell's use of this term.)

in beings with our laws of thought, of a world of diverse but interrelated things' (*EFG*, §58, p. 62). Thus, while rejecting any commitment to Euclidian space that might be present in Kant's version of the argument, Russell endorses a core feature of the argument that is central to Kant's account of the role of intuition in grounding mathematics. With Kant, Russell thinks that because we are conscious of a diversity of things, we must have some form of externality, like Kant's form of outer intuition—space—that non-inferentially grounds in presentations the consciousness of things that are exterior to one another. That is, we must have some non-inferential element of externality in some of our cognitive states, where this externality, as he goes on to explain, must mean 'the fact of Otherness, the fact of being different from some other thing: It must involve the distinction between different things, and must be that element, in a cognitive state, which leads us to discriminate constituent parts in its object' (*EFG*, §58, p. 62).

§5 – The homogeneity of forms of externality

(ii) A form of externality also does not merely give us a heap of indeterminate positions. Rather, if these mutually external positions are to found a geometry, then they must be interrelated (*EFG,* §107, p. 119). A central feature of how Russell conceives of this interrelation, however, is that merely changing the position of a figure should not change any of its intrinsic properties. A space where such a change leaves the intrinsic properties of the figure the same exhibits what Russell refers to as 'the relativity of position'. Russell takes the relativity of position to entail that the parts of space are qualitatively or intrinsically identical, and this closely related feature of space he calls its 'homogeneity'. Russell captures the relativity of position or the homogeneity of space and its parts in his first projective axiom: 'I. We can distinguish different parts of space, but all parts are qualitatively similar, and are distinguished only by the immediate fact that they lie outside one another' (*EFG*, §122, p. 132).

Although ultimately Russell's account of homogeneity needs to be made more precise, as Poincaré shows in his review of *Foundations* (1899),[14] I think we can see that it is a very natural extension of the Kantian account of homogeneity, as allowing for specific identity with numerical difference. According to Russell's first axiom, because positions are qualitatively similar and are distinguished only by their lying outside one another, they are intrinsically

[14] I return to this in note 18 below.

identical. This, Russell maintains, entails 'the homogeneity of space, or its equivalent, the relativity of position' and is required for the possibility of projective transformations (*EFG,* §124, p. 133). Things get more complicated here, but I think we can see what he is thinking. The chief aim of projective geometry, according to Russell, is 'the determination of qualitative spatial similarity' (*EFG,* §123, p. 133). Two figures exhibit such qualitative spatial similarity when, excluding quantitative features like the lengths of their sides, the only thing that distinguishes one figure from another is that they are external to one another. Russell makes this more precise through distinguishing between internal and external relations and properties. On this view, 'it is assumed that a figure can be completely defined by its internal relations' (*EFG*, §124, p. 133). Its external relations, however, 'constitute its position'. External relations can distinguish it from other figures, but a change in its external properties can 'in no way affect its internal properties' (*EFG*, §124, p. 133).

This lines up with Kant's own conception of the internal or intrinsic properties of a figure. According to Kant the intrinsic properties of a figure will include all of the relations among its parts, considered apart from its relations to other figures. For example, the internal properties of a figure will include its dimensionality, its shape, whether its lines are curved or straight, the degree of its angles, and even the ratio of its sides to one another.[15] The extrinsic properties of a figure will be those features that it has in virtue of its relations to other figures outside itself, and this will include the position of the figure.

To go beyond the similarity between Russell and Kant's conceptions of intrinsic properties and see why Russell's account of homogeneity is a natural adaptation of Kant's, we will need to introduce how Russell conceives of the difference between projective and metric geometry. Projective geometry only studies properties of figures that are invariant over projective transformations between possible spaces. These spaces can be of a different curvature and can have different metrics. The metrical distance between two points or the angles of a triangle, say, will not be invariant properties. Still, even if distances and angles are not invariant, what we now call 'cross-ratios' or what Russell called 'anharmonic ratios' are

[15] Compare, e.g., *P,* §13, 4:285-286; *MFNS*, 4:484, A272/B328; Sutherland, 2021 ch. 8. Kant also thinks the direction of a figure is an internal property. This, however, is an important hard case, because it shows that how a figure relates to its space should be included in its internal properties, and so exhibits that a space is prior to the figures in it.

invariant[16] and the fundamental theorem of projective geometry depends on these. What is important for our purposes is that it is only once we move from projective to metric geometry that we consider not merely the ratios between quantitative features, like distances, but the quantitative features of points, lines, and planes, themselves.

In line with this, on Russell's view we can treat the same form of externality and the space it generates at two different levels of abstraction, arriving at either a given metric geometry or at projective geometry. One way that Russell will bring the relationship between these into sharper focus is by pointing out that 'all quantitative comparison presupposes an identity of quality' (*EFG,* §107, p. 119) and that distance, because it 'is a quantitative relation', 'as such presupposes identity of quality' (EFG, §37, p. 34). In this way, according to Russell, to compare two quantities, the quantities must be had by the same kind of thing. For example, it is only if two points lie on the same line that their metrical distance can be measured, the length of a line can only be compared with the length of another line, or it is only areas that can be added to areas. A line cannot be added to an area, because the difference in their dimensionality is a qualitative, specific difference. In this sense Russell is picking up on Kant's adaptation of Euclid's account of the importance of sameness of quality.

In Russell's conception of the homogeneity of space, however, he goes beyond Kant's requirement that two magnitudes or figures must be qualitatively identical to be compared. For Kant, working in uniform Euclidian space, all that was requisite to be able to compose two figures and have this give consistent results across cases was shared dimensionality. In the wake of Riemann and Klein, however, shared dimensionality is insufficient to rule out the possibility that the positions or motions of figures could change their intrinsic properties because they could exist in spaces of non-constant curvature. One natural way to get rid of this complication, however, is to make explicit the requirement that rigid motion through a space cannot distort, or in any way change, the intrinsic properties of a figure.[17] Thus, because

[16] Russell explains anharmonic-ratios this way: 'if through any four points in a straight line we draw four lines which meet in a point, and if we then draw a new straight line meeting these four, the four new points of intersection have the same anharmonic ratio as the four points we started with' (*EFG*, §111-3, p. 122-126). See Gandon (ch. 1, p. 23-24) for another explanation that situates the notion within the history of projective geometry relevant for understanding Russell.

[17] Joseph Delboeuf (1860) puts forward an adaptation of the Kantian account of homogeneity that is stricter than Russell's because it also rules out spaces of non-zero curvature. This is because for Delboeuf a space only counts as homogeneous if increasing or decreasing the length of a figure's sides does not require a corresponding change to the degree of its angles. For discussion, see Jonathan Fay (2024).

this is what Russell is claiming of a space when he claims it is homogeneous, his notion of homogeneity is a natural adaptation of Kant's. [18]

§6 – Why bare numerical difference grounds geometry for Russell

Beyond Russell's adaptation of Kant's account of homogeneity, I think we can see (iii) that forms of externality also serve the same fundamental role of presenting qualitatively identical, but numerically distinct figures. Like Kant's space, we have seen (i) that a form of externality will give a manifold or multiplicity of positions and (ii) that these positions will be systematically interrelated. Now, according to Russell, which relations between positions we consider will determine whether a geometry is projective or metric. According to *Foundations*, projective geometry abstracts away from the quantitative relations and features of positions

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[18] Although Russell only has one conception of homogeneity that he deploys in both metric and projective Geometry, Poincaré shows that he should have had a distinct notion of homogeneity for each. He does this by pointing to an ambiguity in Russell's discussion of internal and external relations and properties.

We are about to see that projective geometry abstracts away from the quantitative relations and features of positions and figures, and studies only their qualitative features, while metric geometry also studies quantitative features like distance. The ambiguity in Russell's discussion of internal and external relations and properties stems from the *extent to which* the space of a form of externality must be homogenous. Within projective geometry, is it okay if changing position effects other non-qualitative relations and properties, so long as it preserves the properties and relations studied by projective geometry? For the homogeneity requisite for projective geometry, after all, changes in other properties will be irrelevant.

Russell can seem to think that the homogeneity required extends farther. This is because when he turns to metric geometry, he seems to hold that we can rule out, *a priori*, that physical space is like the surface of an egg. This is because merely through changes in position in such a space there would be distortions in physical bodies that should only be due to physical causes. His argument hinges on appealing to the homogeneity of space. He takes homogeneity as already given and claims to establish the axiom that '[s]patial magnitudes can be moved from place to place without distortion' on its basis (*EFG*, §144, p. 150). Thus, he seems to hold that homogeneity is a general property of space, not a property relative to a given science, like projective geometry.

Still, this cannot be Russell's considered view. He holds that what is *a priori* is relative to a given science and is whatever renders its subject matter possible (*EFG*, §57, p. 60). The only homogeneity that he has argued is required for projective geometry is one that ensures changing the position of figures in space will not change their inner qualitative properties and relations as they are studied in projective geometry. Thus, because a change in the position of figures in egg space only changes their metric and not these projective properties and relations, it does not seem that egg space should count as heterogeneous in projective geometry. Russel's confusion on this point is apparent in his argument that the homogeneity of space or the relativity of position are required for the possibility of projective transformations (EFG, §123-125, p. 132-4).

This is closely connected to the ambiguity that Poincaré pointed out in Russell's notion of 'qualitative similarity' and 'the relativity of position' (1899, §5). Poincaré asks whether an ellipse and a hyperbola are qualitatively similar or not? Projective geometry would hold that they are, while an everyday, and a metric, conception of quality would hold they are not. Poincaré rightly takes this to show that the notions of 'qualitative similarity' and the 'relativity of position' mean something different in the projective and metric contexts. Thus, Russell should have held that in projective geometry the internal properties of a figure are those properties that are invariant over projective transformations, while external properties are those that change with such transformations. And he should have held that in metrical geometry the internal properties of a figure are those properties preserved through rigid motions, while the external properties are those that are not, like position.

and figures, and studies only their qualitative features (*EFG*, §37, p. 33-34). For example, any two points will define a line. Two points lying on the same line will share a quality, and the line defined by these two points 'may be regarded as a relation of the two points, or an adjective [i.e., property] of the system formed by both together' (*EFG,* §108, p. 120). The 'quantitative' features that will be abstracted away from will have to do with measurement and will include the distance relation between our two points.

Notice first, however, that in the projective context, because it abstracts away from the quantitative features of measurement, knowledge of bare numerical difference with specific identity becomes even more important. To see why, take a one-dimensional form of externality like time and represent it as a line. If this were our only form of externality, and we were to treat the points on this line merely projectively, Russell holds that then the points on this line could not be conceptually distinguished. This is because although they can be distinguished by immediate intuition, 'when we endeavour, without quantity, to distinguish them conceptually, we find the task impossible, since the only qualitative relation of any two of them, the straight line, is the same for any other two' (*EFG*, §121, p. 131).

This issue remains when Russell adds a second dimension. For, suppose we now take another point that does not lie on the line. Then, according to Russell, each of our original points will acquire new qualities, 'namely their relations to the new point, i.e. the straight lines joining them to this new point' (*EFG*, §121, p. 131). All of these lines, however, are qualitatively similar, and they together are what define the new point. 'If we take some other external point, therefore, and join it to the same points of our original straight line, we obtain a figure in which, so long as quantity is excluded, there is no conceptual difference from the former figure' (*EFG*, §121, p. 131). So, as Russell puts it:

> Immediate intuition can distinguish the two figures, but qualitative [i.e., conceptual] discrimination cannot do so […] [T]he only reason, within projective Geometry, for not regarding projective figures as actually identical, is the intuitive perception of difference of position. This is fundamental, and must be accepted as a datum. It is presupposed in the distinction of various points, and forms the very life of Geometry. It is, in fact, the essence of the notion of a form of externality, which notion forms the subject-matter of projective Geometry. (*EFG*, §121, p. 131).

Russell's thought here is the same as the one from Kant above. If we attempted to use only, say, the concept of a line to distinguish and then compose two such figures, we would represent them as instances of the same concept. Then, however, so long as we excluded the use of intuition, the instances would be indistinguishable and hence identical. Thus, we would only succeed in picking out the same thing twice, not in composing two spaces into a new space. In this way, like Kant, Russell seems to see the task of geometry as fundamentally to compare and compose diverse but homogeneous elements, and because this is grounded in a form of externality, projective geometry presupposes such a form.

The case of metric geometry will be similar. In it, we will also consider quantitative determinations, but this will only add more determinations. We saw with Kant that we can easily imagine two numerically distinct, but qualitatively and quantitatively identical raindrops or cubes of space. Such cases—say of one-inch cubes—belong to metric geometry. And so although in metric geometry we have more information with which to distinguish figures, there will still be cases of distinct figures where there is no specific difference between them.

It is thus because forms of intuition and forms of externality ground the very same kind of numerical difference, despite specific identity, that Kant and Russell both put them at the foundation of geometry.

§7 – Why *Foundations* can sound logicist, but isn't

In §2 we saw that Jeremy Heis takes Russell in *Foundations* to endorse what we might call 'transcendental logicism'—the view that mathematics (especially geometry) is grounded in transcendental logic (2017, 308-309). We've now seen that Heis is mistaken insofar as he does not see forms of externality as essentially separate from the intellect. But what of the more general claim that Russell is such a 'logicist'?

Heis claims that on Russell's view in *Foundations,* geometry is grounded in transcendental logic. He argues that this is evident because the argument of *Foundations* is a 'transcendental proof' of the axioms of geometry, which shows that they are necessary conditions for empirical judgments and experience or perception. While Heis is right that the main argument of *Foundations* is such a 'transcendental proof', it is misleading to suggest that the resulting position is 'logicist'. Kant is usually taken to be the arch anti-logicist, and although the traditional logicist positions are a bit of a hodge-podge, one thing that is usually taken to unite them is their rejection of appeals to intuition in grounding (parts of) mathematics.[19] As we've now seen, however, the 'logic' that Russell relies on in *Foundations* allows for appeals to

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[19] This unites, say, the otherwise disparate views of Frege (1884), Dedekind (1888, preface), and Russell (1903, §4; also §249, §432-435). Recent discussions of logic and logicism that pick out the banishment of intuition as the thread that unites otherwise quite diverse views include Howard Stein, (1998, p. 813), Demopoulous and Clark (2005, p. 130), and Michael Kremer (2006, p. 185n2). (Anssi Korhonen (2013, ch. 1; and p. 75-77) also comes close to saying this.)

two distinct and irreducible sources of knowledge in accounting for the possibility of experience: the intellect and forms of externality. We have also seen how forms of externality are, in a quite precise way, a successor to Kant's intuition. Therefore, if we call the Russell of *Foundations* a 'transcendental logicist', then in this 'logicism', in accounting for the principles of mathematics, we allow for appeals to a non-intellectual source of knowledge that is like intuition. Further, in light of this, Kant himself would probably count as a 'logicist' since he takes himself to have given an account of how we know the first principles of arithmetic in the 'Axioms of Intuition' (A162/B202-A165/B207), which belongs to Transcendental Logic. Thus, for both reasons the appellation of 'logicist' is misleading with respect to *Foundations*.

Nonetheless, Heis is right that in *Foundations* we find a lot of rhetoric that sounds logicist. The logicism of *Principles of Mathematics* holds (a) that the fundamental concepts of mathematics are definable in terms of logical concepts; (b) that the basic indemonstrable mathematical truths are logical truths; and (c) that all mathematical reasoning is fundamentally logical reasoning (*PoM*, xv; §1, p. 3). In *Foundations* Russell says things that can seem to indicate that he endorses at least (c) and (a). For example, Russell points out

> that projective Geometry is wholly *a priori*; that [(a)] it deals with an object whose properties are logically deduced from its definition, not empirically discovered from data; that its definition, again, is founded on the possibility of experiencing diversity in relation, or multiplicity in unity; and that our whole science, therefore, [(c)] is logically implied in, and deducible from, the possibility of such experience. (*EFG, §139*, p. 146; §142, 149)

Especially with respect to (a)—according to which the concepts of the object of projective geometry depend only on the 'pure intellect'—the evidence is extensive. For example:

> Projective Geometry, in so far as it deals only with the properties common to all spaces, will be found, if I am not mistaken, to be wholly *a priori,* to take nothing from experience, and to have, like Arithmetic, a creature of the pure intellect for its object. (*EFG*, §103, p. 118)

> [W]hat is merely intuitional can change, without upsetting the laws of thought, without making knowledge formally impossible: but what is purely intellectual cannot change, unless the laws of thought should change, and all our knowledge simultaneously collapse. I shall therefore follow Grassmann's distinction in constructing an *à priori* and purely conceptual form of externality. […] [P]rojective Geometry, abstractly interpreted, is the science which he foresaw, and deals with a matter which can be constructed by the pure intellect alone. (*EFG*, §126, p. 135)

In these passages it can sound as though Russell holds that projective geometry relies on a purely conceptual form of externality, and that the object or matter of projective geometry depends only on the 'pure intellect' and is derivable merely from 'the laws of thought'.

Along with passages like the ones just quoted, one might also note that in *Foundations* Russell repeatedly attempts to downplay intuition's role. For example, he claims that non-Euclidean Geometry 'makes no appeal to intuition' (*EFG*, §56, p. 58), or that it is obvious 'that infinite homogeneous Euclidean space is a concept, not an intuition' (*EFG, p.* 61). He will also explicitly separate the notion of a form of externality from intuition, as when he claims that 'externality, to render the scope of the argument wholly logical, must not be left with a sensational or intuitional meaning' (*EFG*, §58, p. 62).

If, as I have been arguing, forms of externality have the irreducible, fundamental role of grounding the knowledge of bare numerical difference that is essential for geometry, then why do we find so many remarks where Russell seems to be downplaying the role of intuition?

One main motivation behind Russell's position is his anti-psychologism. While he claims that there must be some form of externality grounding geometry, and he views this as a logical or epistemic point, he sees it as a psychological question whether such a form is fundamentally 'sensational' or 'intuitional' (*EFG*, §58, p. 62), and he does not want to take a stand on this.

Relatedly, whenever we have not actually sensed something in a given space, Russell seems to call our knowledge of that space 'conceptual'. So although he will often argue that a form of externality is 'not a mere conception' (*EFG*, §181, p. 179; §183, p. 180; §184, p. 181), he will also claim that any form of externality or space where we have not immediately experienced or actually perceived anything in this space is 'wholly conceptual' (*EFG*, §203, p. 194; §207, p. 198). For similar reasons, he thinks that infinite homogeneous Euclidian space is obviously a concept 'invented to explain an intuition', but not itself an intuition (*EFG*, §58, p. 61). After all, we cannot experience Euclidian space in its infinity.

Russell also takes Kantian intuition to be tied to specifically Euclidian space and its geometry. This is why he can claim that non-Euclidean Geometry 'makes no appeal to intuition' (*EFG*, §56, p. 58). He thinks that Euclidian space is in fact the space of our experience and that it is a contingent psychological fact about human sensibility and our world that it happens to be Euclidian. We can and do, nonetheless, study *a priori* other possible forms of externality and spaces. Because these are *a priori* and independent of Euclidian strictures, he will claim that he is following 'Grassmann's distinction in constructing an *a priori* and purely conceptual form of externality' (*EFG*, §126, p. 135).

How then does Russell view the relationship between forms of externality (as the non-intellectual way in which we are immediately presented with numerically diverse but conceptually identical individuals) and our concepts of them? And in what ways is this continuous or discontinuous with how Kant conceives of the relationship between our formal intuition of space and our concept of space?

Here Russell's remarks on Helmholz are helpful. Russell finds Helmholtz' Flatland and Sphereland to be 'fairy-tale analogies of doubtful value' (*EFG*, §93, p. 101; §94, p. 104). This is because he sides with Kantians like Land in maintaining that higher dimensional and non-Euclidian spaces cannot be imagined (*EFG*, §68, 74). Unlike Land, however, he takes this to show that 'the imaginability or non-imaginability of metageometrical spaces' has become unimportant (*EFG*, §68, 74).[20]

In line with the foregoing, Russell's view of the concepts of various forms of externality is continuous with Kant's view of the concept of space insofar as they both maintain that there would not be these concepts if there were not a possible form of externality or intuition underlying them. Furthermore, just as Russell allows for multiple possible forms of externality, Kant allows that there might be other forms of intuition than our own (e.g., B139, B148, B150-1). Nonetheless, whereas Russell allows that we can perfectly well investigate higher dimensional and non-Euclidian spaces, Kant seems to deny this. This is because he holds that since we cannot imagine other forms of intuition besides our own, we cannot form a positive concept of them (e.g., A230/B283, A254-5/B309-10). Russell agrees that we cannot imagine these forms. He, however, thinks that this does not pose a serious problem for forming a positive concept of them or for investigating their geometries.

This is helpful for understanding why Russell claims that (c) *the inferences* in projective and metric geometry are logical and are carried out by the intellect alone. For Kant, geometrical proofs are 'a chain of inferences guided throughout by intuition' (A717/B745) and in mathematics 'all inferences can be immediately drawn from pure intuition' (A782/B811; compare A734-735/B762-763).[21] Russell, however, in moving to 'metageometry' and in abandoning the need for imagination in conceiving of forms of externality, also takes himself to abandon Kant's reliance on intuition in carrying out the inferences of geometry.[22]

[20] Here my reading is in line with Griffin's (1991, ch. 4 n15).

[21] Whether Kant holds that mathematical inferences always require intuition is a topic of some scholarly discussion, however, because Kant also claims that 'a synthetic proposition can of course be comprehended in accordance with the principle of contradiction, but only insofar as another synthetic proposition is presupposed' (B14). In *Principles*, Russell argues that Kant holds that mathematical inference always involves intuition (1903, §434). In the contemporary discussion, this position is taken up by Friedman (1992, ch. 1) and Hogan (2020).

[22] It's not at all clear, however, that Kant would accept that Russell has banished the reliance on intuition in non-Euclidian inferences. After all, Kant has a capacious sense of how we rely on intuition in mathematics, where intuition is as equally involved in the 'symbolic' constructions of algebra (A734/B762; *ID*, 2:278, 2:291), as it is

In light of all of this, in what sense does Russell's account depend upon a form of externality as a non-intellectual source of knowledge? When Russell claims that the (a) concepts of the object of projective geometry depend only on the 'pure intellect' or (c) *the inferences* in projective and metric geometry are logical, we should not interpret him as denying the dependence of these concepts on a non-intellectual form of externality that allows for the immediate perception of difference in position. He sets aside the psychological question of whether this form of externality is 'sensational' or 'intuitional', instead focusing on its epistemic or (transcendental) 'logical' dimension, in that such a form of externality allows for an ultimately non-intellectual source of knowledge. We can abstract, however, from our immediate perception of difference through our specific form of externality, and form concepts of other such forms. These can then be studied in metric geometry, while projective geometry will be the science of the general properties of all such forms. Thus, although Russell views metric and projective geometry as pursued largely independently of our original form of externality, he is not denying that we must have some such original form of externality that grounds our first concept of bare numerical difference. 'Purely conceptual' forms of externality should not be seen as freeing geometry, whether metric or projective, from the need for a source of knowledge of bare numerical distinctness with no specific difference and, as we've seen, properly speaking, such a source cannot lie in the intellect.

Now, having shown that we must have two fundamentally distinct sources of knowledge according to Russell, I want to close by turning to a further reason why Russell would claim that projective geometry, specifically, 'deals with a matter which can be constructed by the pure intellect alone' (*EFG*, §126, p. 135). This reason will not apply to metric geometry, and although we will see that it has to do with the Kantian lineage of Russell's *Foundations* view, we will also see that it illuminates that view's deepest break with Kant.

Projective geometry abstracts away from the quantitative relations of figures and points. It studies the qualitative relations between points, lines, planes, and figures, without yet specifying the metric of the space in which these entities exist, and so without yet specifying their quantitative, metric relations. For this reason, projective geometry is more general than the metric geometry of this or that space, given by this or that specific form of externality. As

in geometry. Even if Kant agrees with Russell that Helmholtz's Flatland is of doubtful value because it is unimaginable, he should still classify the investigations of non-Euclidian spaces as belonging to something like a branch of algebra, with its symbolic constructions (see Shabel (1998)).

a result, projective geometry will hold for every space, no matter their metric differences, and as Russell puts it, in projective geometry we find 'the qualitative substrata of the metrical superstructure' (*EFG*, §107, p. 119).

Here, I think we can see a Kantian reason for Russell's thought that projective geometry is a conceptual science. Concepts for Kant, remember, are general representations (A320/B376-377; *JL*, §1-§6, 9:91-95). Russell's thought seems to be that because projective geometry is more general than this or that specific metric geometry, what projective geometry studies is not intuitions, or even this or that specific form of externality, but concepts. For this reason, when Russell speaks, for example, of lines and points in projective geometry, we might think of these as really general concepts of lines or points, true of the lines and points in any given form of externality, and we might think of this science as really concerned with these general concepts, like <line> or <point>, not intuitions or forms of externality.

One reason that we should take this to be Russell's most fundamental break with Kant is the Leibnizian spirit of Russell's qualitative substrata. For Kant, general features of spaces are derivative. They depend on an original singular representation of the one actual space. And any qualitative features of figures in space that can be thought through a general concept were taken to be grounded on the one original homogeneous space. In a sense Russell acknowledges this point. The concepts of projective geometry all require that there is *some* original form of externality that makes them possible and grounds them. In a different sense, however, Russell reverses Kant's position by treating projective geometry and the qualitative features that it studies as more fundamental than metric geometry and the features distinctive of this or that individual space. In this sense, Russell takes the fundamental dimensions of space to be conceptual, and he takes our knowledge of them to stem from comparing different possible spaces that we construct. In this way, although he, with Kant, endorses a distinction in kind between the representations of our receptive and intellectual faculties, he takes the common elements shared by any possible form of externality to underlie the receptive features of this or that space. This way in which he privileges the *concept* of space or of a form of externality over the one *individual a priori* form of this or that space is already a fundamental shift away from a Kantian view and back towards a Leibnizian one.[23]

[23] This essay originated as my contribution to the conference 'Logic, Truth, and Objecthood: A celebration of Thomas Ricketts' work in the history of analytic philosophy' at the University of Pittsburgh in the Fall of 2022, and I would like to dedicate the essay to Tom, whose mentorship, support, and guidance was a comfort and a boon over many years. In addition, I very much appreciated having had the opportunity to present the essay on

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